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NOTE
A MIXED VERSION OF MENGER'S THEOREM

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An (a, b)-n-fan means a union of n internally disjoint paths. Menger's theorem states that a graph G has an (a, b)-n-fan if and only if G is n-connected between a and b. We show that G contains λ edge-disjoint (a, b)-n-fans if and only if for any k with $k \le 0 \le \min\{n-1, |V(G)|-2\}$ and for any subset X of $V(G) - \{a, b\}$ with cardinality k, $G - X$ is $\lambda(n-k)$-edge-connected between a and b.

1. Introduction and Notation

In this paper, we shall consider finite graphs and digraphs with no loops but with multiple edges being allowed. (When we consider paths and edges in a digraph, the terms "path" and "edge" always refer to a directed path and a directed edge, respectively.) Let G be a graph or a digraph. We write $V(G)$ for the set of vertices of G and $E(G)$ for the set of edges of G. If $X \subset V(G)$ or $E(G)$, then $G - X$ is the subgraph obtained from G by deleting X. An edge of a digraph G joining $x \in V(G)$ to $y \in V(G)$ is represented by (x, y), while an edge of a graph G joining x and y is represented by xy. Let a and b be distinct vertices of G. Then an $a - b$ path is a path from a to b in G. An (a, b)-fan means a union of internally disjoint $a - b$ paths in G and an (a, b)-n-fan means a union of n internally disjoint $a - b$ paths in G. For an integer $n \ge 1$, G is said to be n(resp. n-edge)-connected between a and b if, whenever less than n vertices of $V(G) - \{a, b\}$ (rest. n edges of $E(G)$) are removed, there still exists an $a - b$ path in G. For two subsets X and Y of $V(G)$, $E_G(X, Y)$ denotes the set of edges xy or (x, y) in G such that $x \in X$ and $y \in Y$. For a vertex v of a digraph G, $od_G v := |E_G(v, V(G) - \{v\})|$ and $id_G v := |E_G(V(G) - \{v\}, v)|$. The reader is referred to [1] and [2] for terms not defined here.

Menger's theorem [6] is one of the most fundamental theorems in graph theory. Its vertex version states that a (di)graph G has an (a, b)-n-fan if and only if G is n-connected between a and b, and its edge version states that a (di)graph G has n edge-disjoint $a - b$ paths if and only if G is n-edge-connected between a and b. As a common generalization of those two versions, we prove:

Theorem. *Let G be a multi(di)graph of order at least two, let a and b be distinct vertices of G, and let λ and n be positive integers. Then, there exist λ edge-disjoint (a, b)-n-fans in G if and only if for any k with $0 \le k \le \min\{n-1, |V(G)|-2\}$ and for*

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any subset X of $V(G) - \{a, b\}$ with cardinality k, $G - X$ is $\lambda(n-k)$-edge-connected between a and b.

We now state the lemmas which we need in the proof of this theorem. The first lemma can be found in [4], where it is proved by the vertex-splitting technique.

Lemma 1 (Ford and Fulkerson [4]). *Let D be a multidigraph of order at least two, let a and b be distinct vertices of G, and let λ and n be positive integers. Then, there exist λn edge-disjoint $a-b$ paths in D such that no vertex in $V(D) - \{a, b\}$ is contained in more than λ of these paths if and only if for any k with $0 \le k \le \min\{n-1, |V(D)|-2\}$ and for any subset X of $V(D) - \{a, b\}$ with cardinality k, $D - X$ is $\lambda(n-k)$-edge-connected between a and b.*

We notice that Lemma 1 holds for an undirected graph as well. In order to see this, replace each undirected edge by a pair of oppositely oriented directed edges, apply Lemma 1 and observe that if both (u, v) and (v, u) are used, then those edges can be eliminated, i.e., if P and Q are edge-disjoint $a-b$ paths and $(u, v) \in E(P)$ and $(v, u) \in E(Q)$, then $P \cup Q - \{(u, v), (v, u)\}$ contains two edge-disjoint $a-b$ paths. Thus we have the following lemma.

Lemma 2. *Let G be an undirected multigraph of order at least two, let a and b be distinct vertices of G, and let λ and n be positive integers. Then, there exist λn edge-disjoint $a-b$ paths $P_1, \ldots, P_{\lambda n}$ in G such that no vertex in $V(G) - \{a, b\}$ is contained in more than λ of these paths if and only if for any k with $0 \le k \le \min\{n-1, |V(G)|-2\}$ and for any subset X of $V(G) - \{a, b\}$ with cardinality k, $G - X$ is $\lambda(n-k)$-edge-connected between a and b.*

Lemma 3 (König [5]). *If G is a λ-regular bipartite undirected multigraph without loops, then $E(G)$ is partitioned into λ 1-factors, i.e., $E(G)$ is partitioned into λ subsets $E_1, \ldots, E_\lambda$ where E_i is a set of pairwise independent edges of G and $|E_i| = (|V(G)|)/2$, $1 \le i \le \lambda$.*

2. Proof of the Theorem

As the proofs of the directed and the undirected versions go parallel and the directed case is easier than the undirected case, we here discuss the undirected case only. The necessity of the condition is clear. To prove the sufficiency, let $P_1, \ldots, P_{\lambda n}$ be the $a-b$ paths in G chosen as in Lemma 2. Let G' be the subgraph of G induced by the edge set $\bigcup_{i=1}^{\lambda n} E(P_i)$. Put $V(G') = \{a, b, v_1, \ldots, v_m\}$. Orient the edges of G' so that each P_i becomes a directed $a-b$ path, and let D denote the resulting digraph. We define a digraph F whose vertex set is a disjoint union of two vertex sets $U = \{a', u_1, \ldots, u_m\}$ and $W = \{b', w_1, \ldots, w_m\}$ as follows: we associate an edge (u_i, w_j) with each edge (v_i, v_j) of $D - \{a, b\}$, and associate an edge (a', w_i) (resp. (u_i, b')) with each edge (a, v_i) (resp. (v_i, b)) and an edge (a', b') with each edge ab, so that $|E_D(v_i, v_j)| = |E_F(u_i, w_j)|$, $|E_D(a, v_i)| = |E_F(a', w_i)|$, $|E_D(v_i, b)| = |E_F(u_i, b')|$ and $|E_D(a, b)| = |E_F(a', b')|$. Note that if $(u_i, w_j) \in E(F)$, then $(u_j, w_i) \notin E(F)$ because we may assume that if $(v_i, v_j) \in E(D)$, then $(v_j, v_i) \notin E(D)$. It is clear that

$1 \le id_D v_i = od_D v_i \le \lambda$ for all v_i, $1 \le i \le m$. Put $p_i := id_D v_i = od_D v_i$. By the definition of F, $id_F w_i = od_F u_i = p_i$, $1 \le i \le m$.

With F as just defined, let F' be the digraph obtained from F by adding $\lambda - p_i$ new (multiple) edges joining u_i to w_i for each i, $1 \le i \le m$. Let E be the set of these new edges, i.e., $E := E(F') - E(F)$. Furthermore, we construct a new digraph F'' from F' as follows: Let A be the set of edges of F' incident from a' and let $A_1, \ldots, A_n$ be a partition of A such that $|A_i| = \lambda$, $1 \le i \le n$. For the vertex b', we define B and B_i, $1 \le i \le n$, similarly. Now we delete the vertices a' and b' and add two disjoint vertex sets $\{a_1, \ldots, a_n\}$ and $\{b_1, \ldots, b_n\}$. We then associate an edge (a_i, w_j) with each edge (a', w_j) of A_i and an edge (u_j, b_i) with each edge (u_j, b') of B_i $(1 \le i \le n,\ j \in M)$, and an edge (a_i, b_j) with each edge (a', b') of $A_i \cap B_j$ $(1 \le i, j \le n)$. In the resulting digraph F'' we have $id_{F''} v + od_{F''} v = \lambda$ for all $v \in V(F'')$, and $E(F'') = E_{F''}(\{a_1, \ldots, a_n\} \cup U', \{b_1, \ldots, b_n\} \cup W')$, where $U' := U - \{a'\}$ and $W' := W - \{b\}$. Therefore, disregarding the "direction" of each edge of F'', we may consider the digraph F'' to be a λ-regular bipartite multigraph without loops with partite sets $\{a_1, \ldots, a_n\} \cup U'$ and $\{b_1, \ldots, b_n\} \cup W'$. It follows from Lemma 3 that $E(F'')$ is partitioned into λ subsets $E_1, \ldots, E_\lambda$ where E_i is a set of pairwise independent edges of F'' and $|E_i| = (|V(F'')|)/2$, $1 \le i \le \lambda$.

In order to complete the proof of the theorem, define a mapping $\alpha : E(F'') - E \to E(G)$ by $\alpha((u_i, w_j)) = v_i v_j$, $\alpha((a_i, w_j)) = av_j$ and $\alpha((u_j, b_i)) = bv_j$, and $\alpha((a_i, b_j) = ab$. Since $E(F'') - E$ does not contain both (u_i, w_j) and (u_j, w_i) where $i \ne j$, it follows that $|E_G(v_i, v_j)| \ge |E_{F''}(u_i, w_j)| + |E_{F''}(u_j, w_i)|$. Therefore, we may assume that the mapping α is one-to-one. Since the edge set $E_i - E$ contains n pairwise disjoint subsets $E_{i_1}, \ldots, E_{i_n}$ such that each E_{i_j} is of the form $\{(a_j, w_{j_1}), (u_{j_1}, w_{j_2}), \ldots, (u_{j_{s-1}}, w_{j_s}), (u_{j_s}, b_{t_j})\}$, where $\bigcup_{j=1}^{n} \{b_{t_j}\} = \{b_1, \ldots, b_n\}$, it follows that the subgraph of G induced by the edge set $\alpha(\bigcup_{j=1}^{n} E_{i_j})$, is an (a, b)-n-fan in G. Since the $\alpha(E_i)$ are pairwise disjoint, this means that G contains λ edge-disjoint (a, b)-n-fans. This completes the proof of the theorem. ∎

An undirected multigraph G is said to be (n, λ)-connected between a and b if for any k with $0 \le k \le \min\{n-1, |V(G)| - 2\}$ and for any subset X of $V(G) - \{a, b\}$ with cardinality k, $G - X$ is $\lambda(n-k)$-edge-connected between a and b. A multigraph G is said to be (n, λ)-connected if for any two distinct vertices a and b of $V(G)$, G is (n, λ)-connected between a and b. Note that n-connected graphs are also $(n, 1)$-connected graphs. Dirac [3] proved that any n vertices in an n-connected graph lie on a common cycle. On the other hand, the undirected version of our theorem implies that if G is $(2, \lambda)$-connected, then for any two vertices a and b, there exist λ edge-disjoint cycles each of which contains both a and b. This suggests the following conjecture:

Conjecture. *Let λ and n be positive integers, let G be an (n, λ)-connected (undirected) multigraph of order at least two, and let S be any subset of $V(G)$ such that $|S| = n$. Then in G there exist λ edge-disjoint cycles $C_1, C_2, \ldots, C_\lambda$ such that $S \subseteq V(C_i)$ for all i.*

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